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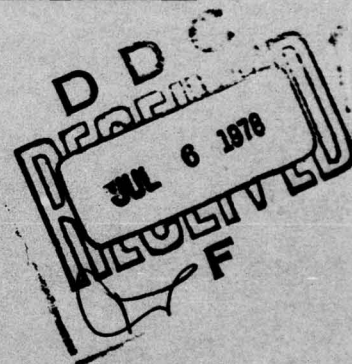ID No.

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In this paper, we demonstrates that the M/M/c queue, with arrival and service rates which vary according to the state of a Markov process, has a steady-state probability vector of a modified matrix-geometric form. The rate matrix R is the unique positive solution to a quadratic matrix equation, which may be solved numerically by successive substitutions. A theorem which provides an accuracy check on that computation is ⟶ next page

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20. Abstract

proved.

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are interpreted.

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FURTHER RESULTS ON THE M/M/1 QUEUE

WITH RANDOMLY VARYING RATES

by

Marcel F. Neuts

University of Delaware
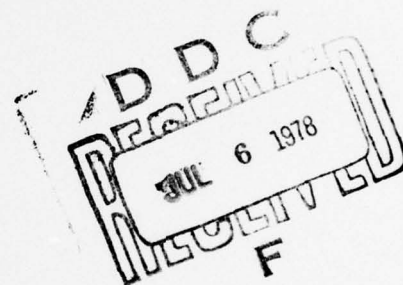
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Abstract

In this paper we show that the M/M/c queue, with arrival and service rates which vary according to the state of a Markov process, has a steady-state probability vector of a modified matrix-geometric form. The rate matrix R is the unique positive solution to a quadratic matrix equation, which may be solved numerically by successive substitutions. A theorem which provides an accuracy check on that computation is proved.

Finally a numerical example is discussed and its results are interpreted.

Key Words

Queueing theory, M/M/c queue, random environment, fluctuating queues, computational probability.

I. Introduction

This paper contains generalizations and further elaborations of results discussed in [1]. We refer to this earlier paper for all definitions and notations. The main new topics of discussion are: the joint stationary probability distribution of the queue length and the underlying phase state, considered at epochs where the phase state changes; the extension of Theorem 5 of [1] to the M/M/c queue; the discussion of some computational aspects and numerical examples. Throughout this paper, we shall only consider stable versions of the queues under discussion.

Lemma 1

Provided that $\underline{\pi}\lambda < \underline{\pi}\mu$, the positive matrix R, which is the unique solution to the equation

$$(1) \qquad R^2 \Delta(\underline{\mu}) + R[Q - \Delta(\underline{\lambda}) - \Delta(\underline{\mu})] + \Delta(\underline{\lambda}) = 0,$$

in the set of nonnegative matrices of spectral radius less than one, satisfies

$$(2) \qquad R\,\underline{\mu} = \underline{\lambda}.$$

Proof

The matrix R was introduced in Theorem 5 of [1]. By postmultiplying (1) by $\underline{e}$, we obtain

$$R^2 \underline{\mu} - R(\underline{\lambda}+\underline{\mu}) + \underline{\lambda} = (I-R)(\underline{\lambda}-R\underline{\mu}) = \underline{0}.$$

Since the spectral radius of R is less than one, the result follows.

<u>Remarks</u>

a. The equality (2) can either be used as an accuracy check on the numerical computation of R or can be incorporated into the evaluation of R, so as to expedite that algorithm.

b. The inner product $\underline{\pi}R\underline{e}$ is easily seen to be the steady-state probability that the server is occupied. It should be noted that in general this quantity is <u>not</u> equal to $\rho = (\underline{\pi}\underline{\lambda})(\underline{\pi}\underline{\mu})^{-1}$. In pedagogical material, it should therefore be stressed that the interpretation given to the traffic intensity is strongly model-dependent.

An exception is the interesting special case, where $\underline{\mu} = \mu^* \underline{e}$. In that case, the equality $\rho = \underline{\pi}R\underline{e}$ follows readily from (2).

II.   The Queue Length at Phase Transitions

The motivation for considering this embedded process is of some interest. As noted in [1], the most interesting cases of the M/M/1 queue in a Markovian environment involve one or more phases j during which $\lambda_j > \mu_j$, even though the global queue is stable. The realizations of such queueing processes exhibit substantial random oscillations which are evident in their effect on the components of the steady-state vector $\underline{x}$ of Thm. 5 [1]. During a sojourn in a phase j for which the queue is "locally" unstable, substantial buildup may occur to be cleared during later phases for which the queue is locally stable.

One might expect to obtain useful additional information by considering the queue lengths at the successive phase transitions. We shall derive the joint stationary probability density of the queue length and the phase immediately before and immediately after phase transitions. We shall however show that the conditional queue length density at the end of a sojourn in the phase j is the same as the conditional queue length density, given that the phase is j. By suitable interpretation of a detailed numerical example in Section IV, we shall see that this result is due to the exponential nature of the sojourn times in the various phases.

It is clear that the sequence $\{(\xi_n, J_n), n \geq 0\}$, where ξ_n is the queue length and J_n the phase state immediately after the n-th phase transition, is a Markov chain on the state space $\{(i,h): i \geq 0, 1 \leq h \leq m\}$. We denote by $p_{ii'}(h,u)$, the probability that in an ordinary M/M/1 queue with parameters λ_h, μ_h, the queue length at time u is i', given that it is i at time 0. Explicit formulas for $p_{ii'}(h,u)$ are known, but will not be needed in our discussion. The transition probability matrix P of the chain $\{(\xi_n, J_n)\}$ is then given by

(3)
$$P_{hh'}(i,i') = \int_0^\infty p_{ii'}(h,u) e^{-\sigma_h u} du \, Q_{hh'},$$

$$\text{for } i \geq 0, \, i' \geq 0, \, h \neq h', \, 1 \leq h, \, h' \leq m.$$

$$= 0, \quad \text{for } i \geq 0, \, i' \geq 0, \, h = h'.$$

The exponent $\sigma_h = -Q_{hh}$, for $1 \leq h \leq m$. We shall find it convenient to partition the matrix P into m×m blocks $P(i,i') = \{P_{hh'}(i,i')\}$, which we write as

(4)
$$P(i,i') = V(i,i')[Q + \Delta(\underline{\sigma})], \quad \text{for } i \geq 0, \, i' \geq 0,$$

where $\Delta(\underline{\sigma}) = \text{diag}(\sigma_1, \ldots, \sigma_m)$ and $V(i,i')$

$$= \text{diag}\left[\int_0^\infty p_{ii'}(h,u) e^{-\sigma_h u} du, \, 1 \leq h \leq m\right].$$

Theorem 1

The stationary probability vector $\underline{y} = (\underline{y}_0, \underline{y}_1, \ldots)$ of P is given by

(5) $\underline{y}_i = (\underline{\pi\sigma})^{-1}\underline{\pi}(I-R)R^i[Q+\Delta(\underline{\sigma})]$, for $i \geq 0$.

Proof

The steady-state equations $\underline{y} = \underline{y}P$, may be written as

(6) $\underline{y}_i = \sum_{i'=0}^{\infty} \underline{y}_{i'}V(i',i)[Q+\Delta(\underline{\sigma})] = \underline{w}_i[Q+\Delta(\underline{\sigma})]$,

for $i \geq 0$, where $\underline{w}_i = \sum_{i'=0}^{\infty} \underline{y}_{i'}V(i',i)$.

Setting $\sum_{i=0}^{\infty} \underline{y}_i = \underline{u}$, we first obtain that

$\underline{u}\Delta^{-1}(\underline{\sigma})[Q+\Delta(\underline{\sigma})]=\underline{u}$. Also $\underline{u}e=1$, since $\underline{y}e=1$. It readily follows

that $\underline{u} = (\underline{\pi\sigma})^{-1}\underline{\pi}\Delta(\underline{\sigma})$, a result which is to be expected.

Setting $\int_0^{\infty} p_{i'i}(h,u)e^{-\sigma_h u}du = p^*_{i'i}(h)$, for $1 \leq h \leq m$, the

birth-and-death equations for the M/M/1 queue yield in a

straightforward manner that

(7) $p^*_{i'0}(h)(\sigma_h+\lambda_h) = \delta_{i'0} + p^*_{i'1}(h)\mu_h$

$p^*_{i'i}(h)(\sigma_h+\lambda_h+\mu_h) = \delta_{i'i} + p^*_{i',i-1}(h)\lambda_h + p^*_{i',i+1}(h)\mu_h$,

for $i' \geq 0$, $i \geq 1$, $1 \leq h \leq m$.

It now readily follows that

(8) $V(i',0)\Delta(\underline{\sigma}+\underline{\lambda}) = \delta_{i'0}I + V(i',1)\Delta(\underline{\mu})$,

$V(i',i)\Delta(\underline{\sigma}+\underline{\lambda}+\underline{\mu}) = \delta_{i'i}I + V(i',i-1)\Delta(\underline{\lambda})$

$+ V(i',i+1)\Delta(\underline{\mu})$,

for $i' \geq 0$, $i \geq 1$.

We further obtain that

$$\underline{w}_0 \Delta(\underline{\sigma}+\underline{\lambda}) = \underline{y}_0 + \underline{w}_1 \Delta(\underline{\mu}) = \underline{w}_0[Q+\Delta(\underline{\sigma})]+\underline{w}_1\Delta(\underline{\mu})$$

$$\underline{w}_i \Delta(\underline{\sigma}+\underline{\lambda}+\underline{\mu}) = \underline{y}_i + \underline{w}_{i-1}\Delta(\underline{\lambda}) + \underline{w}_{i+1}\Delta(\underline{\mu})$$
$$= \underline{w}_i[Q+\Delta(\underline{\sigma})]+\underline{w}_{i-1}\Delta(\underline{\lambda})+\underline{w}_{i+1}\Delta(\underline{\mu}),$$

for $i \geq 0$.

Upon simplification, we notice that the vectors $\underline{w}_i$, $i \geq 0$, satisfy the equations

(10) $$\underline{w}_0[Q-\Delta(\underline{\lambda})] + \underline{w}_1\Delta(\underline{\mu}) = \underline{0}$$

$$\underline{w}_{i-1}\Delta(\underline{\lambda}) + \underline{w}_i[Q-\Delta(\underline{\lambda}+\underline{\mu})] + \underline{w}_{i+1}\Delta(\underline{\mu}) = \underline{0},$$

for $i \geq 1$. These are, except for the normalizing condition, precisely the same equations as satisfied by the vectors $\underline{x}_i$, $i \geq 0$, in Thm. 5 of [1]. Moreover, the vectors $\underline{w}_i$ need to be positive and the sum $\sum_{i=0}^{\infty} \underline{w}_i\underline{e}$ must be finite. It now follows from classical results an irreducible, positive recurrent Markov chains that $\underline{w}_i = k\underline{x}_i$, for $i \geq 0$ and some positive constant k. It follows immediately that $\underline{y}_i = k\underline{\pi}(I-R)R^i[Q+\Delta(\underline{\sigma})]$, for $i \geq 0$, which upon normalization yields the stated result.

Corollary 1

The conditional mean queue length at the beginning of a j-phase is given by

(11) $$\frac{1}{\pi_j\sigma_j}\left\{\underline{\pi}(I-R)^{-1}R[Q+\Delta(\underline{\sigma})]\right\}_j, \quad \text{for } 1\leq j\leq m.$$

The conditional stationary density of the queue length at the beginning of a j-phase is given by

(12) $\qquad \dfrac{1}{\pi_j \sigma_j} \left\{ \underline{\pi}(I-R)R^i[Q+\Delta(\underline{\sigma})] \right\}_j$, $\quad$ for $i \geq 0$.

Corollary 2

Let $\underline{z} = (\underline{z}_0, \underline{z}_1, \ldots)$ with components z_{ij} be the steady-state probability vector of the queue length and the phase immediately before a phase change, then

(13) $\qquad \underline{z}_i = (\underline{\pi\sigma})^{-1}\underline{\pi}(I-R)R^i\Delta(\underline{\sigma})$, $\quad$ for $i \geq 0$.

Proof

The vectors $\underline{z}_i$ and $\underline{y}_i$ are related by

(14) $\qquad \underline{z}_i\Delta^{-1}(\underline{\sigma})[Q+\Delta(\underline{\sigma})] = \underline{y}_i$, $\quad$ for $i \geq 0$.

Clearly the vectors $\underline{z}_i$ in (13) are positive, satisfy the equations (14) and $\underline{z}\underline{e}=1$. This guarantees that $\underline{z}$ is the steady-state vector of the Markov chain, obtained by considering the queue length and phase immediately prior to phase transitions.

Remark

We see that

(15) $\qquad \pi_j^{-1}x_{ij} = (\underline{\pi\sigma})(\pi_j\sigma_j)^{-1}z_{ij}$, $\quad$ for $i \geq 0$, $1 \leq j \leq m$,

so that the conditional probabilities that there are i customers in the queue, given that the phase is j and given that a sojourn in the phase j has ended, are equal.

III. The Multi-server Queue

The M/M/c queue with randomly varying arrival and service rates defines a Markov process with the infinitesimal generator $Q^*(c)$, given by

$$(15) \qquad Q^*(c) = \begin{bmatrix} A_{00} & A_{01} & & & & & \\ A_{10} & A_{11} & A_{12} & & & & \\ & A_{20} & A_{21} & & A_{22} & & \\ & & & A_{c-1,0} & A_{c-1,1} & A_{c-1,2} & \\ & & & & A_0 & A_1 & A_2 \\ & & & & & A_0 & A_1 & A_2 \end{bmatrix},$$

where $A_2 = A_{01} = A_{i2} = \Delta(\underline{\lambda})$, $A_{i0} = i\Delta(\underline{\mu})$, $A_{00} = Q - \Delta(\underline{\lambda})$, and $A_{i,1} = Q - \Delta(\underline{\lambda} + i\underline{\mu})$, for $1 \leq i \leq c-1$. $A_0 = c\Delta(\underline{\mu})$, and $A_1 = Q - \Delta(\underline{\lambda} + c\underline{\mu})$. We assume that $\underline{\lambda} > \underline{0}$ and $\underline{\mu} > \underline{0}$.

Theorem 2

Provided $\rho(c) = (\underline{\pi}\underline{\lambda})(c\underline{\pi}\underline{\mu})^{-1} < 1$, the queue is stable. The steady-state vector $\underline{x} = (\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{c-1}, \underline{x}_c, \ldots)$ is given by

$$(16) \qquad \underline{x}_k = \underline{x}_{c-1} R_c^{k-c+1}, \qquad \text{for } k \geq c.$$

The matrix R_c is the unique solution in the set of non-negative matrices of spectral radius less than one of the

equation

(17) $\quad cR_c^2\Delta(\underline{\mu}) + R_c[Q-\Delta(\underline{\lambda}+c\underline{\mu})] + \Delta(\underline{\lambda}) = 0.$

The matrix R_c is strictly positive and $cR_c\underline{\mu} = \underline{\lambda}.$

The matrix T, given by

(18) $\quad T = \begin{bmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_{11} & A_{12} & & \\ & A_{20} & A_{21} & A_{22} & \\ & & & \diagdown & \\ & & & A_{c-1,0} & A_{c-1,1}+ R_cA_0 \end{bmatrix}$

is an irreducible semi-stable matrix of order cm. The vector $(\underline{x}_0,\underline{x}_1,\ldots,\underline{x}_{c-1})$ is its left eigenvector corresponding to the eigenvalue zero. It is normalized so that

(19) $\quad \underline{x}_0\underline{e} + \ldots + \underline{x}_{c-2}\underline{e} + \underline{x}_{c-1}(I-R_c)^{-1}\underline{e} = 1.$

Proof

The assertions about the matrix R_c were already proved as part of Thm. 5 in [1]. With R_c so chosen, the steady-state equations

$\quad \underline{x}_{i-1}A_0 + \underline{x}_iA_1 + \underline{x}_{i+1}A_2 = \underline{0}, \qquad i\geq c,$

are satisfied, provided $\underline{x}_0,\ldots,\underline{x}_{c-1}$ can be properly chosen.

The c initial equations are equivalent to

$\quad (\underline{x}_0,\ldots,\underline{x}_{c-1})T = \underline{0}.$

Since T is clearly irreducible, it suffices to verify that T is semi-stable. The off-diagonal elements of T are clearly nonnegative. Moreover we have

$$A_{c-1,0}\underline{e} + A_{c-1,1}\underline{e} + R_c A_0 \underline{e} = R_c A_0 \underline{e} - A_{c-1,2}\underline{e}$$

$$= c R_c \underline{\mu} - \underline{\lambda} = \underline{0}.$$

It follows that the diagonal elements of T are negative and hence that T is semi-stable. The normalizing condition (19) uniquely determines $\underline{x}_i$, $0 \leq i < c$.

Remark

It is evident that the proof does not depend on the explicit form of the cm×cm upper left hand corner of $Q^*(c)$, but only on the irreducibility of T. Any variations of the present model which modify only the upper left hand corner will therefore have a steady-state vector of the same general type.

IV. A Numerical Example (c=1)

In order to illustrate the behavior of a queue in which short periods of gross instability alternate with long periods of highly undersaturated conditions, we consider an example where Q is given by the 8×8 matrix

$$Q = \frac{1}{4} \begin{bmatrix} -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

and the vectors $\underline{\lambda}$ and $\underline{\mu}$ are given by

$$\underline{\lambda} = (40, 5, 2, 1, 1, 1, 1, 1),$$

$$\underline{\mu} = (10, 5, 5, 3, 3, 3, 3, 3).$$

The vector $\underline{\pi}$ is given by

$$\underline{\pi} = \frac{1}{29}(1, 4, 4, 4, 4, 4, 4, 4),$$

and $\rho = (\underline{\pi}\underline{\lambda})(\underline{\pi}\underline{\mu})^{-1} = 0.8$.

The equation $R = -A_2A_1^{-1} - R^2A_0A_1^{-1}$, was solved by successive substitutions, which were continued until the

maximum entry-wise difference between iterates was less than 10^{-8}. This took 989 iterations. The vector $R\underline{\mu}$ was evaluated and it was found that $\underline{\lambda}-R\underline{\mu} < 10^{-6}\underline{e}$, with the largest difference occurring in the large first component.

A small improvement in accuracy can be obtained with little additional effort. Let $R'' \leq R'$ be the last two iterates computed, then we compute the matrix R by $R_{jj'} = R'_{jj'} + \theta_j(R'_{jj'} - R''_{jj'})$, where the quantities θ_j are chosen so that $R\underline{\lambda} = \underline{\mu}$.

The conditional means and variances of the queue lengths in the various phases were computed next and are listed below.

Table 1: Conditional Means and Variances in
the Various Phases

Phase	Mean	Variance
1	50.65	2575.55
2	50.98	2581.12
3	41.55	2482.10
4	36.05	2346.10
5	31.29	2178.66
6	27.17	1996.99
7	23.60	1812.20
8	20.50	1631.56

Selected percentiles of the conditional distributions exhibit very clearly the variability of the queue length over the various phases.

Table 2: Deciles of the Conditional Queue Length Distribution

(the first index for which k% is exceeded)

%	Phase 1	Phase 2	Phase 3	Phase 4	Phase 5	Phase 6	Phase 7	Phase 8
10	5	5	0					
20	11	11	2	0	0			
30	18	18	8	2	1	0	0	
40	26	26	16	8	2	1	1	0
50	35	35	25	18	10	4	2	1
60	46	47	36	29	22	14	7	2
70	61	61	51	44	36	29	22	14
80	82	82	71	64	57	50	42	35
90	117	117	107	99	92	85	77	70
95	152	152	142	134	127	120	113	105
99	234	234	223	216	209	202	194	185

Table 3: Conditional Probabilities of Emptiness in the Various Phases

Phase	Probability of Emptiness
1	0.0142
2	0.0168
3	0.1284
4	0.2079
5	0.2701
6	0.3233
7	0.3692
8	0.4090

Interpretation

The numerical results for this example exhibit a number of qualitative features, which deserve to be stressed and interpreted as they are not present in more elementary queueing models.

a. The global mean queue length is given by $\underline{\pi}(I-R)^{-1}R\underline{e} = 33.63$, but this is clearly not a meaningful descriptor of this highly oscillatory queue.

b. The M/M/1 queue with parameters $\lambda^\circ = \underline{\pi}\lambda = \frac{88}{29}$, and $\mu^\circ = \underline{\pi}\mu = \frac{110}{29}$, has a stationary mean queue length of 4, with a variance of 20. It does not begin to offer an approximation to the present queue.

c. The parameters chosen for the numerical example can be thought of as representing a short over-saturated rush hour (Phase 1), a transitional period (Phases 2 and 3) and an unsaturated period (Phases 4 through 8), repeated cyclically. The conditional queue length distributions at the ends of Phases 1,3 and 8 represent the most "extreme" queue conditions. By mixing the conditional queue length densities of the Phases 4, 5,...,8 with the weights $\pi_j(\pi_4+...+\pi_8)^{-1}$, $4\leq j\leq 8$, we obtain the conditional queue length density, given that the queue is in its unsaturated period. This density will clearly be different from that at the end of Phase 8.

d. Although the mean queue lengths vary considerably between phases 1 and 8, there is much less variation in the standard deviations. This is due to two causes. During the oversaturated phase there is enormous random variability in the behavior of the queue. Very long queues at the end of a rush hour will take a long time to dissipate, while shorter queues dissipate quickly. This effect is likely to become apparent in high variances in spite of reduced means in the higher phases.

The second cause of the high variability in those phases lies in the exponential distribution of the sojourn times in each phase. The effect of this assumption is easy to study. By modifying the Q-matrix, we can change the distributions of the rush hour, the transition period and the undersaturated period in a versatile manner without changing their mean durations. These periods can have arbitrary phase type distributions, which may have arbitrary positive coefficients of variation.

In order to illustrate the latter points numerically we computed a number of parametric variants of this model. The results will be presented only briefly, but they are striking indeed.

Variant 1

We replace the exponential duration of the rush-hour by an Erlang distribution of order 2 with parameter 2. This does not affect the mean duration of the rush-hour period, but reduces its variance. The inner products $\underline{\pi}\lambda$ and $\underline{\pi}\mu$ remain as before. There are now nine phases.

Table 4: Conditional Means and Variances in the Various Phases

Phase	Mean	Variance
0	26.69	801.99
1	41.70	1051.32
2	41.87	1077.30
3	32.03	1061.05
4	26.34	985.76
5	21.53	883.98
6	17.53	771.76
7	14.24	660.04
8	11.54	555.57

Phases 0 and 1 now correspond to the rush-hour. We note the drastic reduction in the means and variances in <u>all</u> the phases. The dependence on the random variability of the <u>duration</u> of the rush-hour period is very strong indeed.

Variant 2

In this case we increased the service rate during the transitional period, keeping $\underline{\pi}\mu$ constant by reducing the service rate later in the undersaturated case. The parameters are as in the original model, except for $\underline{\mu}$ which is now (10, 7, 7, 3, 3, 3, 1, 1). The effect of this change in parameters is not as substantial as one might expect.

Table 5: Conditional Means and Variances in
the Various Phases

Phase	Mean	Variance
1	54.56	2775.76
2	47.94	2734.02
3	34.95	2442.26
4	30.49	2258.57
5	26.60	2067.01
6	23.21	1875.74
7	23.98	1846.81
8	24.51	1828.46

Variant 3

The most striking improvement in the conditions of the queue is obtained by increasing the service rate during the rush period itself. This is, of course, not always economically feasible.

The parameters are again as in the original model, except for μ which is now (30, 5, 5, 2, 2, 2, 2, 2).

Table 6: Conditional Means and Variances in
the Various Phases

Phase	Mean	Variance
1	15.75	251.15
2	16.66	260.63
3	10.13	214.98
4	8.49	187.23
5	7.12	161.06
6	5.99	137.07
7	5.07	115.67
8	4.31	97.01

A final comment is concerned with the relationship between the means and corresponding standard deviations in all the preceding examples. We see that queues, which

periodically pass through phases in which they are locally oversaturated, exhibit considerable random variation. To obtain reliable parameter estimates by Monte Carlo simulation, with the rather small sample sizes employed in practice, appears to be impossible. The numerical behavior of the present examples casts serious doubt on the merits of simulation methods, when applied to realistic queueing models, unless prohibitively large sample sizes are used.

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